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TIME RATES OF GENERALIZED STRAIN TENSORS PART I: COMPONENT FORMULAS

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Hill derived a simple component formula for the material time derivative of a generalized Lagrangian strain tensor. We examine Hill's derivation in detail and explain why it is generally valid only when the principal stretches are distinct. We then give a proof of Hill's formula which is valid for any C^2 motion and any C^1 strain measure. Our proof is based on a component form of the chain rule for a tensor-valued function of a time-dependent symmetric tensor. This result is also used to derive component formulas for the Jaumann rate of a generalized Eulerian strain tensor. Finally, we apply the general formulas to the logarithmic strain tensors.

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1 Introduction

Let $\mathcal{V}$ be a three-dimensional real inner product space; for example, $\mathcal{V} = \mathbf{R}^3$ where $\mathbf{R}$ denotes the real numbers. By a *tensor* we mean a linear transformation from $\mathcal{V}$ into $\mathcal{V}$. Let Sym denote the linear space of symmetric tensors, and let Psym denote the open subset of Sym consisting of all symmetric positive-definite tensors. Consider a function $f : \mathbf{R}^+ \to \mathbf{R}$, where $\mathbf{R}^+$ denotes the positive reals. We may define a corresponding isotropic tensor function $\mathbf{f} : \mathrm{Psym} \to \mathrm{Sym}$ as follows. If $\mathbf{A} \in \mathrm{Psym}$ has the spectral decomposition

$$\mathbf{A} = \sum_{i=1}^{3} a_i \mathbf{e}_i \otimes \mathbf{e}_i\,, \tag{1.1}$$

then $\mathbf{f}(\mathbf{A})$ has the spectral decomposition

$$\mathbf{f}(\mathbf{A}) = \sum_{i=1}^{3} f(a_i) \mathbf{e}_i \otimes \mathbf{e}_i\,. \tag{1.2}$$

Note that $\mathbf{A}$ and $\mathbf{f}(\mathbf{A})$ are *coaxial*, i.e., they have a common principal basis $\{\mathbf{e}_i\}$. The eigenvalues of $\mathbf{A}$ and $\mathbf{f}(\mathbf{A})$ corresponding to the eigenvector $\mathbf{e}_i$ are a_i and $f(a_i)$, respectively. In the case where the eigenvalues of $\mathbf{A}$ are not distinct, it is easily shown that the right-hand side of (1.2) is independent of the particular principal basis $\{\mathbf{e}_i\}$ used in the spectral decomposition of $\mathbf{A}$.

Now consider a *motion* $\mathcal{X}$ of a deformable body over some time interval I. We identify the body with a *reference configuration* $\mathcal{B} \subset \mathcal{V}$, so that the motion is a mapping $\mathcal{X} : \mathcal{B} \times I \to \mathcal{V}$. The *deformation gradient* $\mathbf{F} = \mathrm{D}\mathcal{X}$ admits the *polar decomposition*

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}\,, \tag{1.3}$$

where the symmetric positive-definite tensors $\mathbf{U}$ and $\mathbf{V}$ are called the *right* and *left stretch tensors*, respectively, and the proper orthogonal tensor $\mathbf{R}$ is called the *rotation tensor*. The *principal stretches* $\{\lambda_i\}$ are the eigenvalues of $\mathbf{U}$ and $\mathbf{V}$. A principal basis $\{\mathbf{u}_i\}$ of $\mathbf{U}$ is called a *Lagrangian triad*; a principal basis $\{\mathbf{v}_i\}$ of $\mathbf{V}$ is called an *Eulerian triad*. These triads are said to be *corresponding* if

$$\mathbf{v}_i = \mathbf{R}\mathbf{u}_i\,, \quad i = 1,2,3\,. \tag{1.4}$$

Such a correspondence is always possible in view of $(1.3)_2$. We denote the *stretching tensor* by $\mathbf{D}$ and the *spin tensor* by $\mathbf{W}$; they are the symmetric and skew parts, respectively, of the *velocity gradient* $\mathbf{L}$. If Φ is a scalar, vector or tensor field associated with the motion $\mathcal{X}$, then $\dot{\Phi}$ denotes the *material time derivative* of Φ.

By a *strain measure* we mean a suitably smooth function $f : \mathbf{R}^+ \rightarrow \mathbf{R}$ satisfying

$$f(1) = 0\,, \quad f'(1) = 1\,, \quad f' > 0\,. \tag{1.5}$$

Hill [8] introduced the class of *generalized Lagrangian strain tensors*. These are tensors of the form $\mathbf{f}(\mathbf{U})$, where $\mathbf{f}$ is the tensor function corresponding to the strain measure f. For example, by taking $f(x) = (x^2 - 1)/2$ we obtain the *Green-St. Venant strain tensor* (often called the *Lagrangian strain tensor*) $(\mathbf{C} - \mathbf{I})/2$, where $\mathbf{I}$ denotes the identity tensor and $\mathbf{C} \equiv \mathbf{F}^T\mathbf{F} = \mathbf{U}^2$ is the *right Cauchy-Green tensor*. Hill [9,10] derived a simple component formula for the material time derivative of $\mathbf{f}(\mathbf{U})$:

$$f(U)^{\cdot}_{ij} = \begin{cases} \lambda_i f'(\lambda_i)\, D_{ij} & \text{if } \lambda_i = \lambda_j \\ \dfrac{2\lambda_i\lambda_j}{\lambda_i + \lambda_j}\dfrac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j}\, D_{ij} & \text{if } \lambda_i \neq \lambda_j\,. \end{cases} \tag{1.6}$$

Here f' denotes the derivative of f, $\{D_{ij}\}$ are the components of $\mathbf{D}$ relative to the Eulerian triad, and $\{f(U)^{\cdot}_{ij}\}$ are the components of $\mathbf{f}(\mathbf{U})^{\cdot}$ relative to the corresponding Lagrangian triad. We will refer to equation (1.6) as *Hill's Formula*. Hill [9] used this formula to obtain a simple component formula for the stress tensor $\mathbf{S}$ work-conjugate to the strain tensor $\mathbf{f}(\mathbf{U})$ for an arbitrary elastic material. In the same paper Hill utilized (1.6) to study a class of constitutive inequalities of the form

$$\mathrm{tr}(\dot{\mathbf{S}}\mathbf{f}(\mathbf{U})^{\cdot}) > 0 \tag{1.7}$$

for isotropic elastic solids. Apart from these important applications, Hill's Formula is of general interest since most of the finite strain tensors used in continuum mechanics are of the form $\mathbf{f}(\mathbf{U})$ or $\mathbf{f}(\mathbf{V})$ for some analytic strain measure f.

Gurtin and Spear [6] and Hoger [11] remarked that the derivations of Hill's Formula given by Hill and others are not rigorous when the principal stretches are repeated, i.e., when $\lambda_i = \lambda_j$ for some $i \neq j$. In fact, we are not aware of any rigorous proof of Hill's Formula for the case of repeated principal stretches.[1] Nevertheless, Hill's Formula and his proof are widely cited in the mechanics literature without qualification. Hence we feel that both a rigorous proof of Hill's Formula and a detailed explanation of the gaps in Hill's proof are called for. By a rigorous proof we mean one which is valid for any sufficiently smooth motion $\mathcal{X}$ and any sufficiently smooth strain measure f. Throughout this paper we assume only that $\mathcal{X}$ is C^2, i.e.,

[1]For the special case $f = \ln$, a rigorous proof has been given by Hoger [11]; see the discussion at the end of Section 6.

twice continuously differentiable. Then **F**, **R**, **U** and **V** are C^1 functions of position and time, and **L**, **D** and **W** are continuous functions of position and time, where the position may be taken relative to either the reference or the current configuration. As will become clear from the results in Sections 2–4, requiring $\boldsymbol{\mathcal{X}}$ to be C^∞ instead of just C^2 does not lead to any simplification in the proof of Hill's Formula.

In Section 2 we give two simple examples of a C^∞ homogeneous motion for which the Lagrangian and Eulerian triads are discontinuous functions of time. It follows that any proof of Hill's Formula which requires the differentiability of the Lagrangian triads cannot be valid for all C^∞ motions. In this section we also review some relevant theorems on the smoothness of eigenvalues and eigenvectors.

In Section 3 we review the derivation of Hill's Formula given by Hill [10] and we discuss the difficulties with the "limiting process" proposed by Hill for handling the case of repeated principal stretches. We also show that this limiting process may be avoided whenever the Lagrangian triad is a differentiable function of time, and thus in particular whenever the number of distinct principal stretches is constant or whenever **U** is an analytic function of time.

In Section 4 we derive a component form of the chain rule for a tensor-valued function of a time-dependent symmetric tensor. This general result, together with some well-known kinematic identities, yields a simple but rigorous proof of Hill's Formula. In Section 5 we derive analogous component formulas for $\mathbf{f}(\mathbf{V})^\circ$ — the Jaumann rate of the *generalized Eulerian strain tensor* $\mathbf{f}(\mathbf{V})$. We show that these formulas hold relative to any corresponding Lagrangian and Eulerian triads, provided that the strain measure f is C^1. In Section 6 we apply the results of the preceding sections to the logarithmic strain tensors.

In Part II we use the component formulas of the present paper to derive approximate basis-free formulas for $\mathbf{f}(\mathbf{U})^\cdot$ and $\mathbf{f}(\mathbf{V})^\circ$. In Part III we show how the component formulas can be converted to exact basis-free formulas of the type obtained by Hoger [11] for the special case $f = \ln$.

2 Smooth motions with discontinuous Lagrangian triads

A common misconception in the mechanics literature is that smoothness of the Lagrangian triad corresponding to a fixed material point follows from smoothness of the motion. Two counterexamples are given below. For both examples we consider a motion $\mathcal{X} : \mathcal{V} \times \mathsf{R} \to \mathsf{R}$ of the form

$$\mathcal{X}(\mathbf{X}, t) = \mathbf{U}(t)\mathbf{X}, \tag{2.1}$$

where $\mathbf{U} : \mathsf{R} \to \text{Psym}$. Then $\mathbf{F} = \mathbf{U} = \mathbf{V}$ and $\mathbf{R} = \mathbf{I}$. Hence $\mathcal{X}$ is a homogeneous pure stretch; in particular, the Lagrangian and Eulerian triads coincide and are independent of position $\mathbf{X}$.

For the first example, let

$$\mathbf{U}(t) = \begin{cases} \mathbf{I} + e^{-1/t^2}\mathbf{A}^+ & t > 0 \\ \mathbf{I} & t = 0 \\ \mathbf{I} + e^{-1/t^2}\mathbf{A}^- & t < 0, \end{cases} \tag{2.2}$$

where $\mathbf{A}^+$ and $\mathbf{A}^-$ are noncoaxial symmetric positive-definite tensors with eigenvalues $\{a_i^+\}$ and $\{a_i^-\}$, respectively.[2] Then $\mathbf{U}(t)$ has eigenvalues

$$\lambda_i(t) = \begin{cases} 1 + e^{-1/t^2}a_i^+ & t > 0 \\ 1 & t = 0 \\ 1 + e^{-1/t^2}a_i^- & t < 0. \end{cases} \tag{2.3}$$

Since each λ_i is positive, $\mathbf{U}(t) \in \text{Psym}$ for all times t. Both $\mathbf{U}$ and its eigenvalues are C^∞ functions whose derivatives of all orders vanish at time $t = 0$. For $t > 0$ the principal axes of $\mathbf{U}(t)$ are the principal axes of $\mathbf{A}^+$, while for $t < 0$ the principal axes of $\mathbf{U}(t)$ are the principal axes of $\mathbf{A}^-$. Since $\mathbf{A}^+$ and $\mathbf{A}^-$ are noncoaxial, the principal axes of $\mathbf{U}$ have a jump discontinuity at $t = 0$. Hence, for this choice of $\mathbf{U}$ the motion $\mathcal{X}$ is a C^∞ homogeneous pure stretch with C^∞ principal stretches, and yet the Lagrangian triad has a jump discontinuity at time $t = 0$.

For the second example, let

$$\mathbf{U}(t) = \begin{cases} \mathbf{I} + e^{-1/t^2}\mathbf{A}(t) & t \neq 0 \\ \mathbf{I} & t = 0, \end{cases} \tag{2.4}$$

where

$$[\mathbf{A}(t)] = \begin{bmatrix} \cos(2/t) & \sin(2/t) & 0 \\ \sin(2/t) & -\cos(2/t) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.5}$$

relative to some fixed orthonormal basis.[3] Then $\lambda_i(0) = 1$, and for $t \neq 0$ the eigenvalues of $\mathbf{U}(t)$ are

$$\lambda_1(t) = 1 + e^{-1/t^2}, \quad \lambda_2(t) = 1 - e^{-1/t^2}, \quad \lambda_3(t) = 1. \tag{2.6}$$

[2]This is a trivial modification of an example due to H. Shaw in the paper by Goff [4].
[3]This is a trivial modification of an example due to Rellich [19, p. 41].

Since each λ_i is positive, $\mathbf{U}(t) \in$ Psym for all times t. Both $\mathbf{U}$ and its eigenvalues are C^∞ functions whose derivatives of all orders vanish at time $t = 0$. For $t \neq 0$, a corresponding C^∞ principal basis is

$$\begin{aligned} \mathbf{u}_1(t) &= (\cos(1/t), \sin(1/t), 0) \\ \mathbf{u}_2(t) &= (\sin(1/t), -\cos(1/t), 0) \\ \mathbf{u}_3(t) &= (0, 0, 1). \end{aligned} \tag{2.7}$$

These basis vectors are unique to within a sign since the $\lambda_i(t)$ are distinct for $t \neq 0$. Although any orthonormal basis is a principal basis for $\mathbf{U}(0)$, the basis vectors $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ do not have limits as $t \to 0^-$ or $t \to 0^+$. In fact, if we set

$$\mathbf{e}_1(\theta) = (\cos\theta, \sin\theta, 0) \quad \text{and} \quad \mathbf{e}_2(\theta) = (\sin\theta, -\cos\theta, 0), \tag{2.8}$$

then for any angle θ there is a sequence of times $t_n \to 0$ such that $\mathbf{u}_1(t_n) = \mathbf{e}_1(\theta)$ and $\mathbf{u}_2(t_n) = \mathbf{e}_2(\theta)$ for each positive integer n; e.g., take $t_n = 1/(\theta + 2n\pi)$. Also note that

$$\dot{\mathbf{u}}_1(t) = \frac{1}{t^2}\mathbf{u}_2(t), \quad \dot{\mathbf{u}}_2(t) = -\frac{1}{t^2}\mathbf{u}_1(t). \tag{2.9}$$

Hence, for this choice of $\mathbf{U}$ the motion $\boldsymbol{\mathcal{X}}$ is a C^∞ homogeneous pure stretch with C^∞ principal stretches, and yet the Lagrangian triad is discontinuous at time $t = 0$ and spins at a rate which becomes infinite as $t \to 0$.

The two examples above have the following properties in common:

1. The stretch tensor $\mathbf{U}$ is not an analytic function of time at $t = 0$.

2. The multiplicity of some of the principal stretches changes at $t = 0$; i.e., there is no time interval containing $t = 0$ on which the principal stretches have constant multiplicity.

3. The principal stretches are not distinct at $t = 0$.

Indeed, if the stretch tensor $\mathbf{U}$ corresponding to a fixed material point is a C^1 function of time, and if $\mathbf{U}$ fails to have a C^1 principal basis at time $t = 0$, then these conditions must hold. This is a consequence of the following fundamental theorems:

1. *An analytic time-dependent symmetric tensor has analytic eigenvalues and a corresponding analytic principal basis.*

2. *A C^k $(k = 1, 2, \ldots, \infty)$ time-dependent symmetric tensor has C^k eigenvalues and a corresponding C^k principal basis on any time interval for which the the eigenvalues have constant multiplicity, i.e., on any time interval for which the number of distinct eigenvalues is constant.*

3. *The eigenvalues of a continuous time-dependent symmetric tensor are also continuous functions when ordered by magnitude.*

These theorems are valid for any finite-dimensional inner product space. The first theorem is due to F. Rellich; proofs can be found in Rellich [19, §1.1] and Kato [12, §II.6]. Proofs of the second theorem can be found in Nomizu [17][4] and Kato [12, §II.5.7, §II.4.2]. A proof of the third theorem can be found in Rellich [19, §1.3]. One consequence of this theorem is that if the eigenvalues of a continuous symmetric tensor $\mathbf{A}$ are distinct at some time t, then $\mathbf{A}$ has continuous distinct eigenvalues on some time interval containing t; if $\mathbf{A}$ is in fact C^k ($k = 1, 2, \ldots, \infty$), then by the second theorem the eigenvalues and corresponding principal basis are C^k at time t.

Requiring the principal stretches to have constant multiplicity, or requiring the stretch tensor to be an analytic function of time, is too restrictive in general. Consider, for example, a smooth wave traveling into a region of the body which is at rest. Let $\mathbf{X}_0$ be a material point which is ahead of the wave for all times $t < t_0$ and on the wavefront at $t = t_0$. Since $\mathbf{U}(\mathbf{X}_0, t)$ is constant for $t < t_0$, $\mathbf{U}$ must be a non-analytic function of time at $t = t_0$, for otherwise $\mathbf{U}$ would be constant for all time.

3 Hill's Formula for smooth Lagrangian triads

We begin by reviewing the derivation of Hill's Formula given by Hill [10]. Let $\hat{\mathbf{D}}$ denote the *rotated stretching tensor*:

$$\hat{\mathbf{D}} \equiv \mathbf{R}^{\mathrm{T}}\mathbf{D}\mathbf{R}, \tag{3.1}$$

where a T superscript denotes the transpose. Recall the well-known formula (Truesdell and Noll [20])

$$\hat{\mathbf{D}} = \frac{1}{2}(\dot{\mathbf{U}}\mathbf{U}^{-1} + \mathbf{U}^{-1}\dot{\mathbf{U}}), \tag{3.2}$$

which follows from the polar decomposition of $\mathbf{F}$ and the identity $\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}$. The right stretch tensor $\mathbf{U}$ has the spectral decomposition

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{u}_i \otimes \mathbf{u}_i \,. \tag{3.3}$$

Let $\{\dot{U}_{ij}\}$ and $\{\hat{D}_{ij}\}$ denote the components of $\dot{\mathbf{U}}$ and $\hat{\mathbf{D}}$, respectively, relative to the Lagrangian triad $\{\mathbf{u}_i\}$; for example,

$$\dot{\mathbf{U}} = \sum_{i,j=1}^{3} \dot{U}_{ij} \mathbf{u}_i \otimes \mathbf{u}_j \,, \quad \dot{U}_{ij} = \mathbf{u}_i \cdot \dot{\mathbf{U}} \mathbf{u}_j \,, \tag{3.4}$$

[4] Nomizu states his result for the C^∞ case only, but his proof is valid for the C^k case.

where $\mathbf{a} \cdot \mathbf{b}$ denotes the inner product of the vectors $\mathbf{a}$ and $\mathbf{b}$. Let $\{D_{ij}\}$ denote the components of the stretching tensor $\mathbf{D}$ relative to the corresponding Eulerian triad $\{\mathbf{v}_i\}$. Since $\mathbf{v}_i = \mathbf{R}\mathbf{u}_i$, the component form of (3.1) and (3.2) is (Hill [10])

$$\hat{D}_{ij} = D_{ij} = \frac{\lambda_i + \lambda_j}{2\lambda_i \lambda_j} \dot{U}_{ij} \,. \tag{3.5}$$

This formula is valid relative to any corresponding Lagrangian and Eulerian triads; in particular, the triads need not be continuous functions of time.

Now assume that the Lagrangian triad $\{\mathbf{u}_i\}$ corresponding to any fixed material point is a differentiable function of time. Since $\lambda_i = \mathbf{u}_i \cdot \mathbf{U}\mathbf{u}_i$, the principal stretches are also differentiable functions of time. Let $\bar{\mathbf{\Omega}}$ denote the spin of the Lagrangian triad, i.e., $\bar{\mathbf{\Omega}}$ is the time-dependent skew tensor satisfying

$$\bar{\mathbf{\Omega}}\mathbf{u}_i = \dot{\mathbf{u}}_i \,. \tag{3.6}$$

By differentiating the spectral decomposition (3.3) and using (3.6), we obtain

$$\dot{\mathbf{U}} = \sum_{i=1}^{3} \dot{\lambda}_i \mathbf{u}_i \otimes \mathbf{u}_i + \bar{\mathbf{\Omega}}\mathbf{U} - \mathbf{U}\bar{\mathbf{\Omega}}. \tag{3.7}$$

The generalized Lagrangian strain tensor $\mathbf{f}(\mathbf{U})$ corresponding to the strain measure f has the spectral decomposition

$$\mathbf{f}(\mathbf{U}) = \sum_{i=1}^{3} f(\lambda_i) \mathbf{u}_i \otimes \mathbf{u}_i \,. \tag{3.8}$$

By differentiating (3.8) and using (3.6), we obtain

$$\mathbf{f}(\mathbf{U})^{\cdot} = \sum_{i=1}^{3} f'(\lambda_i) \dot{\lambda}_i \mathbf{u}_i \otimes \mathbf{u}_i + \bar{\mathbf{\Omega}}\mathbf{f}(\mathbf{U}) - \mathbf{f}(\mathbf{U})\bar{\mathbf{\Omega}} \,. \tag{3.9}$$

Let $\{\bar{\Omega}_{ij}\}$ and $\{f(U)^{\cdot}_{ij}\}$ denote the components of $\bar{\mathbf{\Omega}}$ and $\mathbf{f}(\mathbf{U})^{\cdot}$, respectively, relative to $\{\mathbf{u}_i\}$. Then the component form of (3.7) and (3.9) is

$$\dot{U}_{ij} = \begin{cases} \dot{\lambda}_i & \text{if } i = j \\ (\lambda_j - \lambda_i)\bar{\Omega}_{ij} & \text{if } i \neq j \,, \end{cases} \tag{3.10}$$

and

$$f(U)^{\cdot}_{ij} = \begin{cases} f'(\lambda_i)\dot{\lambda}_i & \text{if } i = j \\ \left[f(\lambda_j) - f(\lambda_i)\right] \bar{\Omega}_{ij} & \text{if } i \neq j \,. \end{cases} \tag{3.11}$$

By substituting (3.10) into (3.5), we obtain (Hill [9,10])

$$D_{ij} = \begin{cases} \dot{\lambda}_i / \lambda_i & \text{if } i = j \\ \dfrac{\lambda_j^2 - \lambda_i^2}{2\lambda_i \lambda_j} \bar{\Omega}_{ij} & \text{if } i \neq j \,. \end{cases} \tag{3.12}$$

By solving $(3.12)_1$ for $\dot{\lambda}_i$ and solving $(3.12)_2$ for $\bar{\Omega}_{ij}$, and then substituting the results into (3.11), we obtain (Hill [9,10])

$$f(U)^{\cdot}_{ii} = \lambda_i f'(\lambda_i)\, D_{ii}\,, \tag{3.13}$$

and

$$f(U)^{\cdot}_{ij} = \frac{2\lambda_i\lambda_j}{\lambda_i+\lambda_j}\frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}\, D_{ij} \quad \text{if } i \neq j \text{ and } \lambda_i \neq \lambda_j\,. \tag{3.14}$$

It remains to determine $f(U)^{\cdot}_{ij}$ for the case where $i \neq j$ and $\lambda_i = \lambda_j$. Hill [9] claimed that the formula

$$f(U)^{\cdot}_{ij} = \lambda_i f'(\lambda_i)\, D_{ij} \quad \text{if } i \neq j \text{ and } \lambda_i = \lambda_j \tag{3.15}$$

follows "by a limiting process", but he did not provide any details. Hill [10] stated that the coefficient of D_{ij} in (3.14) approaches $\lambda_i f'(\lambda_i)$ in the limit as $\lambda_i \to \lambda_j$. While this is indeed true, it does not constitute a proof of (3.15), as we will explain below. Equations (3.13)-(3.15) constitute Hill's Formula — equation (1.6). For the case of distinct principal stretches, various authors have re-derived Hill's Formula using essentially the same arguments given above. Some authors (Guo and Dubey [5]) have also invoked Hill's "limiting process" for the case of repeated principal stretches.

Let us examine Hill's proof more closely. Recall that the following assumptions were utilized:

1. The Lagrangian triad $\{\mathbf{u}_i\}$ corresponding to a fixed material point is a differentiable function of time.

2. The function f is differentiable.

Clearly, Hill's derivation of equations (3.13) and (3.14) is valid under assumptions 1 and 2. In particular, f need not be a strain measure. Hence for the remainder of this paper we do not impose the conditions (1.5) on f. Moreover, since $\mathbf{U}$ is a C^1 function we know from the discussion in §2 that, for a fixed material point, λ_i and $\mathbf{u}_i$ $(i = 1,2,3)$ are in fact C^1 functions of time at any instant at which the λ_i are distinct. And since (3.13) and (3.14) constitute Hill's Formula in this case, it follows that for any point $\mathbf{X}$ and time t at which the three principal stretches are distinct, Hill's derivation of Hill's Formula is valid under assumption 2 only. Now consider a point $\mathbf{X}_0$ and time t_0 at which $\lambda_i(\mathbf{X}_0, t_0) = \lambda_k(\mathbf{X}_0, t_0)$ for some $i \neq k$. The examples in §2 show that the Lagrangian triad need not be a continuous, let alone differentiable, function of time at $t = t_0$. Therefore, in general we cannot eliminate assumption 1 in Hill's derivation of (3.13) and (3.14).

Now assume that $i \neq j$ and consider Hill's derivation of (3.15) from (3.14). One problem with Hill's limiting argument is that we are not free to allow λ_i to approach λ_j in (3.14). For a given motion $\mathcal{X}$, the principal stretches are given functions of position and time, and (3.14) is an identity which holds only for those points $\mathbf{X}$ and times t at which $\lambda_i(\mathbf{X},t) \neq \lambda_j(\mathbf{X},t)$. If $\lambda_i(\mathbf{X}_0,t) = \lambda_j(\mathbf{X}_0,t)$ for all t in some time interval containing t_0, then (3.14) does not hold for any time t in this interval. Clearly, in this case no information about $f(U)^{\cdot}_{ij}(\mathbf{X}_0,t_0)$ can be obtained from (3.14) by taking limits as $t \to t_0$. Similarly, if $\lambda_i(\mathbf{X},t_0) = \lambda_j(\mathbf{X},t_0)$ for all points $\mathbf{X}$ in some neighborhood of $\mathbf{X}_0$, then no information about $f(U)^{\cdot}_{ij}(\mathbf{X}_0,t_0)$ can be obtained from (3.14) by taking limits as $\mathbf{X} \to \mathbf{X}_0$. Therefore Hill's derivation of (3.15) from (3.14) "by a limiting process" is not valid even if assumption 1 holds. Summarizing, we conclude that *Hill's derivation of Hill's Formula is generally valid only for those points* $\mathbf{X}$ *and times* t *at which the three principal stretches are distinct, in which case his derivation is valid assuming only that the strain measure* f *is differentiable.*

There is a trivial but rigorous proof of (3.15) which avoids any limiting process and which is valid under assumptions 1 and 2. By $(3.11)_2$ and $(3.12)_2$ we see that

$$f(U)^{\cdot}_{ij} = D_{ij} = 0 \quad \text{if } i \neq j \text{ and } \lambda_i = \lambda_j \,. \tag{3.16}$$

Thus (3.15) holds trivially! This simple argument appears to have been overlooked in the literature. Since (3.13) and (3.14) have also been shown to hold under assumptions 1 and 2, we have shown that *Hills' formula is valid for all times under assumptions 1 and 2.* We have also observed that assumption 1 can be eliminated when the principal stretches are distinct. In fact for a given material point, assumption 1 can be eliminated on any time interval during which the number of distinct principal stretches is constant. For on any such time interval the Lagrangian triad is a C^1 function of time; see the second theorem at the end of §2. Similarly, from the first theorem at the end of §2 we see that Hill's Formula is valid whenever $\mathbf{U}$ is an analytic function of time.

For times t_0 at which the multiplicity of some of the principal stretches changes, it turns out that Hill's Formula can be obtained from (3.13) and (3.14) by a rigorous limiting process even when assumption 1 is eliminated, but this result is by no means obvious. The proof, which requires the additional assumption that both f and the corresponding tensor function $\mathbf{f}$ are C^1, will be omitted in favor of the simple proof of Hill's Formula given in the next section. Here we merely point out one of the technical difficulties which must be addressed when a limiting process is employed. As shown by the examples in §2, the Lagrangian triad $\{\mathbf{u}_i\}$ and the Eulerian triad $\{\mathbf{v}_i\}$ need not have limits as $t \to t_0$. Thus, even though $\mathbf{f}(\mathbf{U})^{\cdot}$ and $\mathbf{D}$ are continuous, the components $f(U)^{\cdot}_{ij} = \mathbf{u}_i \cdot \mathbf{f}(\mathbf{U})^{\cdot}\mathbf{u}_j$ and $D_{ij} = \mathbf{v}_i \cdot \mathbf{D}\mathbf{v}_j$ in (3.13) and (3.14) need not have limits as $t \to t_0$.

4 A rigorous proof of Hill's Formula

Our proof of Hill's Formula is based on a component form of the chain rule for a tensor-valued function of a time-dependent symmetric tensor. Consider any differentiable function $f : \mathbf{R}^+ \to \mathbf{R}$, and let $\mathbf{f}$: Psym $\to$ Sym denote the corresponding tensor function. Let $\mathbf{A} : I \to$ Psym be differentiable, where I is an interval of $\mathbf{R}$. Then

$$\mathbf{f}(\mathbf{A}) \equiv \mathbf{f} \circ \mathbf{A} : I \to \mathrm{Sym}. \tag{4.1}$$

We interpret I as a time interval, so that $\mathbf{A}$ and $\mathbf{f}(\mathbf{A})$ are interpreted as time-dependent symmetric tensors. Although these tensors need not be associated with the motion of a body, we will denote their derivatives by $\dot{\mathbf{A}}$ and $\mathbf{f}(\mathbf{A})^{\cdot}$, respectively. By the chain rule, $\mathbf{f}(\mathbf{A})$ is differentiable and

$$\mathbf{f}(\mathbf{A})^{\cdot}(t) = \mathrm{D}\mathbf{f}(\mathbf{A}(t))[\dot{\mathbf{A}}(t)] \tag{4.2}$$

at each time $t \in I$. $\mathrm{D}\mathbf{f}(\mathbf{A}(\mathrm{t}))$, the derivative of $\mathbf{f}$ at the point $\mathbf{A}(t) \in$ Psym, is a linear transformation from Sym into Sym and thus may be regarded as a fourth order tensor. Let

$$\mathbf{A}(t) = \sum_{i=1}^{3} a_i(t)\mathbf{e}_i(t) \otimes \mathbf{e}_i(t) \tag{4.3}$$

be any time-dependent spectral decomposition of $\mathbf{A}$. Let $\{f(A)^{\cdot}_{ij}(t)\}$, $\{\dot{A}_{kl}(t)\}$ and $\{\mathcal{F}_{ijkl}(t)\}$ denote the components of $\mathbf{f}(\mathbf{A})^{\cdot}(t)$, $\dot{\mathbf{A}}(t)$ and $\mathrm{D}\mathbf{f}(\mathbf{A}(\mathrm{t}))$, respectively, relative to the principal basis $\{\mathbf{e}_i(t)\}$. Then the component form of (4.2) is

$$f(A)^{\cdot}_{ij}(t) = \sum_{k,l=1}^{3} \mathcal{F}_{ijkl}(t)\dot{A}_{kl}(t). \tag{4.4}$$

It can be shown (see below) that

$$\begin{aligned}
\mathcal{F}_{iiii}(t) &= f'(a_i(t)), && (4.5)\\
\mathcal{F}_{ijij}(t) &= \mathcal{F}_{ijji}(t) \\
&= \begin{cases} \dfrac{1}{2}\dfrac{f(a_i(t)) - f(a_j(t))}{a_i(t) - a_j(t)} & \text{if } a_i(t) \neq a_j(t) \\[2ex] \dfrac{1}{2}f'(a_i(t)) \quad \text{if } a_i(t) = a_j(t) \text{ and } i \neq j, \end{cases} && (4.6)\\
\mathcal{F}_{ijkl}(t) &= 0 \quad \text{if } \{i,j\} \neq \{k,l\}. && (4.7)
\end{aligned}$$

When (4.5)–(4.7) are substituted into (4.4), we obtain

$$f(A)^{\cdot}_{ij}(t) = \begin{cases} f'(a_i(t))\,\dot{A}_{ij}(t) & \text{if } a_i(t) = a_j(t) \\ \dfrac{f(a_i(t)) - f(a_j(t))}{a_i(t) - a_j(t)}\,\dot{A}_{ij}(t) & \text{if } a_i(t) \neq a_j(t). \end{cases} \tag{4.8}$$

This is the component formula on which our proof of Hill's Formula is based. We emphasize that no smoothness restrictions on the principal basis $\{\mathbf{e}_i\}$ are required here; in particular, $\{\mathbf{e}_i\}$ need not be a continuous function of time.

The component formulas (4.5)–(4.7) have been obtained by several authors under various conditions. For analytic f see Hausner [7] and Kenney and Laub [13]. Bowen and Wang [2] and Chadwick and Ogden [3] derived component formulas for the derivative of an arbitrary isotropic tensor function. Their derivations are repeated in the books by Wang and Truesdell [21, §6.4] and Ogden [18, §6.1.4]. Their results reduce to (4.5)–(4.7) when applied to the isotropic tensor function $\mathbf{f}$ corresponding to the scalar function f. The formulas (4.5)–(4.7) are also stated on p. 162 of Ogden's book. None of these authors state precisely the conditions under which their formulas hold. However, it is not hard to show that their proofs, when specialized to the case under consideration here, are valid under the assumption that both f and $\mathbf{f}$ are C^1. This raises a question which was not addressed by any of the authors above. Can $\mathbf{f}$ fail to be C^1 if f is C^1?

We claim that the smoothness of the scalar function f and the corresponding tensor function $\mathbf{f}$ are related as follows:

1. If f is C^1 then $\mathbf{f}$ is C^1.

2. If f is C^{k+1} $(k = 2, 3, \ldots)$ then $\mathbf{f}$ is C^k.

3. If f is C^∞ then $\mathbf{f}$ is C^∞.

4. If $\mathbf{f}$ is C^k $(k = 1, 2, \ldots, \infty)$ then f is C^k.

Of course, the third result follows from the second. The fourth result follows easily from definition (1.2). The first and second results will be established below. We do not know whether "f is $C^k \Rightarrow \mathbf{f}$ is C^k" is true for $k \geq 2$.

With these results in hand, we can now give a simple but rigorous proof of Hill's Formula, assuming only that the function $f : \mathbf{R}^+ \to \mathbf{R}$ is C^1. Since the corresponding tensor function $\mathbf{f}$ is necessarily C^1, and since the right stretch tensor $\mathbf{U}$ is a C^1 function of position and time, it follows that $\mathbf{f}(\mathbf{U})$ is a C^1 function of position and time, where position may be taken relative to either the reference or the current configuration. By letting $\mathbf{A} = \mathbf{U}$ in (4.8), we obtain

$$f(U)^{\cdot}_{ij} = \begin{cases} f'(\lambda_i)\,\dot{U}_{ij} & \text{if } \lambda_i = \lambda_j \\ \dfrac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j}\,\dot{U}_{ij} & \text{if } \lambda_i \neq \lambda_j\,, \end{cases} \tag{4.9}$$

relative to any Lagrangian triad. Now recall that the component formula (3.5) holds relative to any corresponding Lagrangian and Eulerian triads. By solving (3.5) for $\dot{U}_{ij}$ and substituting the result into (4.9), we see that *Hill's Formula, equation* (1.6), ***holds relative to any corresponding Lagrangian and Eulerian triads if f is C^1.***

It remains to establish 1 and 2 above. Recall that a function $\sigma : \text{Psym} \to \mathbf{R}$ is isotropic iff there is a symmetric function $\hat{\sigma} : (\mathbf{R}^+)^3 \to \mathbf{R}$ such that

$$\sigma(\mathbf{A}) = \hat{\sigma}(a_1, a_2, a_3)\,, \tag{4.10}$$

where $\{a_i\}$ are the eigenvalues of $\mathbf{A}$ (here $\mathbf{A}$ denotes a fixed but arbitrary symmetric positive-definite tensor). Ball [1] has shown that σ is C^2 if $\hat{\sigma}$ is C^2, and that σ is C^r ($r = 3, 4, \ldots$) if $\hat{\sigma}$ is C^{r+1}. If we set

$$\hat{\sigma}(x_1, x_2, x_3) \equiv \sum_{i=1}^{3} \int_1^{x_i} f\,, \tag{4.11}$$

then $\hat{\sigma}$ is obviously symmetric, and $\hat{\sigma}$ is C^{r+1} if f is C^r. We claim that the tensor function $\mathbf{f}$ corresponding to f is the gradient of the isotropic function σ corresponding to $\hat{\sigma}$:

$$\mathbf{f}(\mathbf{A}) = \nabla\sigma(\mathbf{A})\,, \quad \forall\, \mathbf{A} \in \text{Psym}\,. \tag{4.12}$$

Then 1 and 2 above follow from the Ball's results; for example, f is $C^1 \Rightarrow \hat{\sigma}$ is $C^2 \Rightarrow \sigma$ is $C^2 \Rightarrow \nabla\sigma$ is $C^1 \Rightarrow \mathbf{f}$ is C^1. To prove (4.12), first note that $\nabla\sigma : \text{Psym} \to \text{Sym}$ is isotropic since σ is isotropic; hence every principal basis of $\mathbf{A}$ is a principal basis of $\nabla\sigma(\mathbf{A})$. If $\mathbf{A} = \sum_{i=1}^{3} a_i \mathbf{e}_i \otimes \mathbf{e}_i$ is a spectral decomposition of $\mathbf{A}$, it follows that

$$\nabla\sigma(\mathbf{A}) = \sum_{i=1}^{3} \alpha_i \mathbf{e}_i \otimes \mathbf{e}_i \tag{4.13}$$

for some $\{\alpha_i\}$. Then, for example,

$$\begin{aligned} \alpha_1 &= \mathbf{e}_1 \cdot \nabla\sigma(\mathbf{A})\mathbf{e}_1 = \nabla\sigma(\mathbf{A}) \cdot (\mathbf{e}_1 \otimes \mathbf{e}_1) \\ &= \frac{d}{dt}\sigma(\mathbf{A} + t\mathbf{e}_1 \otimes \mathbf{e}_1)\bigg|_{t=0} = \frac{d}{dt}\hat{\sigma}(a_1 + t, a_2, a_3)\bigg|_{t=0} \\ &= \frac{d}{dt}\int_1^{a_1+t} f\,\bigg|_{t=0} = f(a_1)\,. \end{aligned}$$

Similarly, $\alpha_2 = f(a_2)$ and $\alpha_3 = f(a_3)$. Then (4.13) and (1.2) yield (4.12).

5 Component formulas for the rates of $\mathbf{f}(\mathbf{V})$

Since the left stretch tensor has the spectral decomposition

$$\mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i \,, \tag{5.1}$$

$\mathbf{f}(\mathbf{V})$ has the spectral decomposition

$$\mathbf{f}(\mathbf{V}) = \sum_{i=1}^{3} f(\lambda_i) \mathbf{v}_i \otimes \mathbf{v}_i \,. \tag{5.2}$$

The class of *generalized Eulerian strain tensors* consists of all tensors of the form (5.2) for some strain measure f. Hill [8,9,10] confined his analysis to the class of generalized Lagrangian strain tensors. Both classes of strain tensors and their rates have been discussed by Wang and Truesdell [21], Nemat-Nasser [15,16] and Ogden [18]. Here, as in the previous section, we assume that $f : \mathbf{R}^+ \to \mathbf{R}$ is any C^1 function. Then $\mathbf{f}(\mathbf{V})$ is also C^1.

From (5.2), (3.8) and (1.4), it follows that

$$\mathbf{f}(\mathbf{V}) = \mathbf{R}\mathbf{f}(\mathbf{U})\mathbf{R}^{\mathrm{T}} \,. \tag{5.3}$$

Let $\mathbf{\Omega}$ denote the spin of the rotation tensor $\mathbf{R}$:

$$\mathbf{\Omega} \equiv \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} = -\mathbf{R}\dot{\mathbf{R}}^{\mathrm{T}} = -\mathbf{\Omega}^{\mathrm{T}} \,. \tag{5.4}$$

Since $\mathbf{v}_i = \mathbf{R}\mathbf{u}_i$, $\mathbf{\Omega}$ may be interpreted as the relative spin of the Eulerian and Lagrangian triads. However, even when these triads are discontinuous functions of time, $\mathbf{\Omega}$ is continuous since $\mathbf{R}$ is C^1. From the polar decomposition and the identity $\dot{\mathbf{F}} = \mathbf{L}\,\mathbf{F}$, we find that $\mathbf{\Omega}$ and the spin tensor $\mathbf{W}$ are related by the following formula (Truesdell and Noll [20]):

$$\mathbf{W} = \mathbf{\Omega} + \frac{1}{2}\mathbf{R}(\dot{\mathbf{U}}\mathbf{U}^{-1} - \mathbf{U}^{-1}\dot{\mathbf{U}})\mathbf{R}^{\mathrm{T}}. \tag{5.5}$$

It will be convenient to introduce the following *corotational rates* of a tensor field $\mathbf{\Phi}$:

$$\mathbf{\Phi}^{\circ} \equiv \dot{\mathbf{\Phi}} + \mathbf{\Phi}\mathbf{W} - \mathbf{W}\mathbf{\Phi} \tag{5.6}$$

and

$$\mathbf{\Phi}^{*} \equiv \dot{\mathbf{\Phi}} + \mathbf{\Phi}\mathbf{\Omega} - \mathbf{\Omega}\mathbf{\Phi} \,. \tag{5.7}$$

$\mathbf{\Phi}^{\circ}$ is usually called the *Jaumann rate* of $\mathbf{\Phi}$. By taking $\mathbf{\Phi} = \mathbf{f}(\mathbf{V})$ in the above, we see that

$$\begin{aligned} \mathbf{f}(\mathbf{V})^{\cdot} &= \mathbf{f}(\mathbf{V})^{\circ} + \mathbf{W}\mathbf{f}(\mathbf{V}) - \mathbf{f}(\mathbf{V})\mathbf{W} \\ &= \mathbf{f}(\mathbf{V})^{*} + \mathbf{\Omega}\mathbf{f}(\mathbf{V}) - \mathbf{f}(\mathbf{V})\mathbf{\Omega} \,. \end{aligned} \tag{5.8}$$

By taking the material time derivative of (5.3) and using (5.4) and $(5.8)_2$, we obtain

$$\mathbf{f(V)}^{*} = \mathbf{R}\mathbf{f(U)}^{\cdot}\mathbf{R}^{\mathrm{T}}. \tag{5.9}$$

Let $\{f(V)^{\cdot}{}_{ij}\}$, $\{f(V)^{\circ}{}_{ij}\}$, $\{f(V)^{*}{}_{ij}\}$, $\{\Omega_{ij}\}$ and $\{W_{ij}\}$ denote the components of $\mathbf{f(V)}^{\cdot}$, $\mathbf{f(V)}^{\circ}$, $\mathbf{f(V)}^{*}$, $\mathbf{\Omega}$ and $\mathbf{W}$, respectively, relative to the Eulerian triad $\{\mathbf{v}_i\}$. Then the component form of (5.8) is

$$\begin{aligned} f(V)^{\cdot}{}_{ij} &= f(V)^{\circ}{}_{ij} - [f(\lambda_i) - f(\lambda_j)]\,W_{ij} \\ &= f(V)^{*}{}_{ij} - [f(\lambda_i) - f(\lambda_j)]\,\Omega_{ij}\,, \end{aligned} \tag{5.10}$$

relative to any Eulerian triad. The component form of (5.9) is

$$f(V)^{*}{}_{ij} = f(U)^{\cdot}{}_{ij}\,, \tag{5.11}$$

relative to any corresponding Lagrangian and Eulerian triads. Then Hill's Formula yields a formula for $f(V)^{*}{}_{ij}$ in terms of D_{ij}; by substituting that formula into $(5.10)_2$, we obtain a formula for $f(V)^{\cdot}{}_{ij}$ in terms of D_{ij} and Ω_{ij}, relative to any Eulerian triad. In particular, from (5.10), (5.11) and Hill's Formula, we see that relative to any corresponding Lagrangian and Eulerian triads,

$$\begin{aligned} f(V)^{\cdot}{}_{ij} &= f(V)^{\circ}{}_{ij} = f(V)^{*}{}_{ij} \\ &= f(U)^{\cdot}{}_{ij} = \lambda_i f'(\lambda_i)\,D_{ij} \quad \text{if } \lambda_i = \lambda_j\,. \end{aligned} \tag{5.12}$$

The component form of (5.5) is (Hill [10])

$$W_{ij} = \Omega_{ij} + \frac{\lambda_i - \lambda_j}{2\lambda_i\lambda_j}\,\dot{U}_{ij}\,, \tag{5.13}$$

relative to any corresponding Lagrangian and Eulerian triads. By solving (3.5) for $\dot{U}_{ij}$ and substituting the result into (5.13), we obtain (Hill [10])

$$\Omega_{ij} = W_{ij} - \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}\,D_{ij}\,, \tag{5.14}$$

relative to any Eulerian triad. Then from (5.10), (5.11), Hill's Formula and (5.14), we obtain

$$f(V)^{\circ}{}_{ij} = \frac{\lambda_i^2 + \lambda_j^2}{\lambda_i + \lambda_j}\,\frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j}\,D_{ij} \quad \text{if } \lambda_i \neq \lambda_j\,, \tag{5.15}$$

relative to any Eulerian triad. This, together with (5.12) and $(5.10)_1$, yields a component formula for $f(V)^{\cdot}{}_{ij}$ in terms of D_{ij} and W_{ij}, relative to any Eulerian triad. Also, from (5.15), (5.12),(5.11) and Hill's Formula, we obtain

$$f(V)^{\circ}{}_{ij} = \frac{\lambda_i^2 + \lambda_j^2}{2\lambda_i\lambda_j}\,f(V)^{*}{}_{ij}\,, \tag{5.16}$$

relative to any Eulerian triad.

That the above component formulas for the rates of $\mathbf{f}(\mathbf{V})$ hold for any C^2 motion and any C^1 function f is established here for the first time. For distinct principal stretches, these formulas can be obtained from the component formulas in Nemat-Nasser [15,16].

Note that while our formulas for $f(V)^{*}{}_{ij}$ and $f(V)^{\circ}{}_{ij}$ involve components relative to the Eulerian triad only, our derivation of these formulas made use of Hill's Formula, which involves components relative to the Lagrangian triad. We now show how these formulas can be derived without recourse to Hill's formula. From the identity $\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}$, we obtain

$$\dot{\mathbf{B}} = \mathbf{B}\mathbf{D} + \mathbf{D}\mathbf{B} + \mathbf{W}\mathbf{B} - \mathbf{B}\mathbf{W}\,, \tag{5.17}$$

where $\mathbf{B} \equiv \mathbf{F}\mathbf{F}^{\mathrm{T}} = \mathbf{V}^2$ is the *left Cauchy-Green tensor.* Let $\{\dot{V}_{ij}\}$ denote the components of $\dot{\mathbf{V}}$ relative to the Eulerian triad. Since $\dot{\mathbf{B}} = \dot{\mathbf{V}}\mathbf{V} + \mathbf{V}\dot{\mathbf{V}}$, the component form of (5.17) is equivalent to

$$\dot{V}_{ij} = \frac{\lambda_i^2 + \lambda_j^2}{\lambda_i + \lambda_j}\, D_{ij} - (\lambda_i - \lambda_j)\, W_{ij}\,, \tag{5.18}$$

relative to any Eulerian triad. From (4.8) with $\mathbf{A} = \mathbf{V}$, we have

$$f(V)^{\cdot}{}_{ij} = \begin{cases} f'(\lambda_i)\, \dot{V}_{ij} & \text{if } \lambda_i = \lambda_j \\ \dfrac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j}\, \dot{V}_{ij} & \text{if } \lambda_i \neq \lambda_j\,, \end{cases} \tag{5.19}$$

relative to any Eulerian triad. By substituting (5.18) into (5.19) and using $(5.10)_1$, we obtain (5.15) and $(5.12)_{1,4}$. From these results, (5.10) and (5.14), we find that $f(V)^{*}{}_{ij}$ is given by the right-hand side of Hill's Formula and that (5.16) holds.

We can also derive the formulas for $f(V)^{\circ}{}_{ij}$ as follows. Let $\mathbf{h}$ denote the tensor function corresponding to the C^1 function $h : \mathbf{R}^+ \to \mathbf{R}$. By setting $f = h$, $\mathbf{A} = \mathbf{B}$ and $a_i = b_i \equiv \lambda_i^2$ in (4.8), and then using the identity (5.17), we obtain

$$h(B)^{\cdot}{}_{ij} = \begin{cases} 2b_i h'(b_i)\, D_{ij} & \text{if } b_i = b_j \\ (b_i + b_j)\dfrac{h(b_i) - h(b_j)}{b_i - b_j}\, D_{ij} - [h(b_i) - h(b_j)]\, W_{ij} & \text{if } b_i \neq b_j\,, \end{cases} \tag{5.20}$$

where $\{h(B)^{\cdot}{}_{ij}\}$ denote the components of $\mathbf{h}(\mathbf{B})^{\cdot}$ relative to the Eulerian triad. If we now set $h(x^2) = f(x)$, then $\mathbf{h}(\mathbf{B}) = \mathbf{f}(\mathbf{V})$ and thus $h(B)^{\cdot}{}_{ij} = f(V)^{\cdot}{}_{ij}$. Hence, the formulas $(5.12)_{1,4}$ and (5.15) follow from (5.20) and $(5.10)_1$.

6 The logarithmic strain tensors

In this section we consider the generalized strain tensors $\mathbf{f}(\mathbf{U})$ and $\mathbf{f}(\mathbf{V})$ corresponding to the logarithmic strain measure $f = \ln$. To be consistent with our scheme of notation these tensors should be denoted by $\ln(\mathbf{U})$ and $\ln(\mathbf{V})$. However, it is customary to denote the *logarithmic strain tensors* by $\ln \mathbf{U}$ and $\ln \mathbf{V}$, and we will follow this practice here.

The importance of the logarithmic strain tensors is due in part to their simple relation to the stretching tensor for certain simple motions. For example, for a pure stretch ($\mathbf{R} \equiv \mathbf{I}$) with fixed Lagrangian triads, we have $\mathbf{V} = \mathbf{U}$ and

$$(\ln \mathbf{V})^{\cdot} = (\ln \mathbf{V})^{\circ} = (\ln \mathbf{V})^{*} = (\ln \mathbf{U})^{\cdot} = \mathbf{D}. \tag{6.1}$$

By "fixed Lagrangian triads" we mean that the Lagrangian triad corresponding to a given material point $\mathbf{X}$ is fixed, i.e., time-independent, but we allow the possibility that the triad may vary with $\mathbf{X}$. If we assume only that the Lagrangian triads are fixed, then since their spin $\bar{\mathbf{\Omega}}$ is zero the formulas in §3 and §5 imply that $D_{ij} = 0$ for $i \neq j$, and

$$\mathbf{W} = \mathbf{\Omega} \equiv \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}, \tag{6.2}$$

$$(\ln \mathbf{U})^{\cdot} = \dot{\mathbf{U}}\mathbf{U}^{-1} = \mathbf{U}^{-1}\dot{\mathbf{U}} = \hat{\mathbf{D}}, \tag{6.3}$$

$$(\ln \mathbf{V})^{*} = (\ln \mathbf{V})^{\circ} = \mathbf{V}^{\circ}\mathbf{V}^{-1} = \mathbf{V}^{-1}\mathbf{V}^{\circ} = \mathbf{D}, \tag{6.4}$$

$$(\ln \mathbf{V})^{\cdot} = \mathbf{D} + \mathbf{W}(\ln \mathbf{V}) - (\ln \mathbf{V})\mathbf{W}. \tag{6.5}$$

When $\mathbf{R} \equiv \mathbf{I}$, we recover (6.1). The formulas (6.2)–(6.5) are actually valid under conditions slightly weaker than the condition of fixed Lagrangian triads; see Gurtin and Spear [6] and Hoger [11]. None of the formulas above hold in general, as can be seen from the component formulas below.

For the remainder of this section we assume only that the motion $\boldsymbol{\mathcal{X}}$ is C^2. From (5.12) we obtain the component formulas

$$(\ln V)^{\cdot}{}_{ij} = (\ln V)^{\circ}{}_{ij} = (\ln V)^{*}{}_{ij} = (\ln U)^{\cdot}{}_{ij} = D_{ij} \quad \text{if } \lambda_i = \lambda_j. \tag{6.6}$$

Let

$$\Lambda_{ij} \equiv \lambda_i / \lambda_j \tag{6.7}$$

and

$$l_{ij} \equiv \ln \Lambda_{ij} = \ln \lambda_i - \ln \lambda_j = -l_{ji}. \tag{6.8}$$

Then from (5.11), Hill's Formula and (5.15), we obtain the following component formulas when $\lambda_i \neq \lambda_j$:

$$\begin{aligned}(\ln V)^{*}{}_{ij} = (\ln U)^{\cdot}{}_{ij} &= \frac{2\lambda_i\lambda_j}{\lambda_i + \lambda_j}\,\frac{\ln \lambda_i - \ln \lambda_j}{\lambda_i - \lambda_j}\,D_{ij} \\ &= \frac{2\Lambda_{ij}\ln \Lambda_{ij}}{\Lambda_{ij}^2 - 1}\,D_{ij} = \frac{l_{ij}}{\sinh l_{ij}}\,D_{ij}\end{aligned} \tag{6.9}$$

and

$$
\begin{aligned}
(\ln V)^{\circ}{}_{ij} &= \frac{\lambda_i^2+\lambda_j^2}{\lambda_i+\lambda_j}\,\frac{\ln\lambda_i-\ln\lambda_j}{\lambda_i-\lambda_j}\,D_{ij} \\
&= \frac{\Lambda_{ij}^2+1}{\Lambda_{ij}^2-1}(\ln\Lambda_{ij})\,D_{ij} = \frac{l_{ij}}{\tanh l_{ij}}\,D_{ij}\,. \qquad (6.10)
\end{aligned}
$$

Component formulas for $(\ln V)^{\cdot}{}_{ij}$ when $\lambda_i \neq \lambda_j$ follow from (6.7)–(6.10) and the component formula

$$
(\ln V)^{\cdot}{}_{ij} = (\ln V)^{\circ}{}_{ij} - l_{ij}W_{ij} = (\ln V)^{*}{}_{ij} - l_{ij}\Omega_{ij}\,, \qquad (6.11)
$$

which follows from (5.10). These component formulas are valid relative to any corresponding Lagrangian and Eulerian triads. Note that all of the components depend on the principal stretches only through their ratios, and thus are unaffected by the dilatational part of the deformation. For $(\ln U)^{\cdot}{}_{ij}$ this was first observed by Hill [9,10]. In fact, by (5.12) we see that this condition characterizes the logarithmic strain measure. That is, the logarithmic strain measure is the only strain measure f such that, for all C^2 motions, the components of any one of the tensors $\mathbf{f}(\mathbf{U})^{\cdot}$, $\mathbf{f}(\mathbf{V})^{\cdot}$, $\mathbf{f}(\mathbf{V})^{\circ}$ or $\mathbf{f}(\mathbf{V})^{*}$ depend on the principal stretches only through their ratios.

The last expression for $(\ln U)^{\cdot}{}_{ij}$ in (6.9) has been noted by Mehrabadi and Nemat-Nasser [14]. The other formulas for $(\ln U)^{\cdot}{}_{ij}$ are due to Hill [9,10]. The derivations given by these authors are valid for the case of three distinct principal stretches. For the other cases, the first rigorous proof of the formulas for $(\ln U)^{\cdot}{}_{ij}$ is due to Hoger [11]. Hoger derived basis-free formulas for $(\ln \mathbf{U})^{\cdot}$ and then claimed (without providing additional details) that those formulas imply Hill's component formulas. For the case of three distinct principal stretches, Hoger's basis-free formula is so complicated that a verification of this claim would involve some horrendous algebraic manipulations. On the other hand, it is only when some of the principal stretches are repeated that Hill's derivation breaks down, and for these cases Hoger's basis-free formulas are sufficiently simple that the corresponding component formulas can be obtained without much effort.[5]

[5]**Equation $(5.11)_2$ for the coefficient Θ_2 in Hoger's [11] formula for $(\ln \mathbf{U})^{\cdot}$ contains a misprint—the minus sign should be applied only to the first term in the numerator, not to the entire right-hand side.**

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References

[1] J. M. Ball. Differentiability properties of symmetric and isotropic functions. *Duke Mathematical Journal*, 51:699–728, 1984.

[2] R. M. Bowen and C.-C. Wang. Acceleration waves in inhomogeneous isotropic elastic bodies. *Archive for Rational Mechanics and Analysis*, 38:13–45, 1970.

[3] P. Chadwick and R. W. Ogden. A theorem of tensor calculus and its application to isotropic elasticity. *Archive for Rational Mechanics and Analysis*, 44:54–68, 1971.

[4] S. Goff. Hermitian function matrices which commute with their derivative. *Linear Algebra and its Applications*, 36:33–40, 1981.

[5] Z.-H. Guo and R. N. Dubey. Basic aspect's of Hill's method in solid mechanics. *Solid Mechanics Archives*, 9:353–380, 1984.

[6] M. E. Gurtin and K. Spear. On the relationship between the logarithmic strain rate and the stretching tensor. *International Journal of Solids and Structures*, 19:437–444, 1983.

[7] M. Hausner. Eigenvalues of certain operators on matrices. *Communications on Pure and Applied Mathematics*, 14:155–156, 1961.

[8] R. Hill. On constitutive inequalities for simple materials—I. *Journal of the Mechanics and Physics of Solids*, 16:229–242, 1968.

[9] R. Hill. Constitutive inequalities for isotropic elastic solids under finite strain. *Proceedings of the Royal Society of London, Series A*, 314:457–472, 1970.

[10] R. Hill. Aspects of invariance in solid mechanics. In C.-S. Yih, editor, *Advances in Applied Mechanics*, volume 18, pages 1–75. Academic Press, New York, 1978.

[11] A. Hoger. The material time derivative of logarithmic strain. *International Journal of Solids and Structures*, 22:1019–1032, 1986.

[12] T. Kato. *A Short Introduction to Perturbation Theory for Linear Operators.* Springer-Verlag, New York, 1982.

[13] C. Kenney and A. J. Laub. Condition estimates for matrix functions. *SIAM Journal on Matrix Analysis and Applications*, 10:191–209, 1989.

[14] M. M. Mehrabadi and S. Nemat-Nasser. Some basic kinematical relations for finite deformations of continua. *Mechanics of Materials*, 6:127–138, 1986.

[15] S. Nemat-Nasser. On finite-deformation elasto-plasticity. *International Journal of Solids and Structures*, 18:857–872, 1982.

[16] S. Nemat-Nasser. On finite plastic flow of crystalline solids and geomaterials. *Journal of Applied Mechanics*, 50:1114–1126, 1983.

[17] K. Nomizu. Characteristic roots and vectors of a differentiable family of symmetric matrices. *Linear and Multilinear Algebra*, 1:159–162, 1973.

[18] R. W. Ogden. *Non-Linear Elastic Deformations*. Ellis Horwood, Chichester, England, 1984.

[19] F. Rellich. *Perturbation Theory of Eigenvalue Problems*. Gordon and Breach, New York, 1969.

[20] C. Truesdell and W. Noll. The Non-Linear Field Theories of Mechanics. In *Encyclopedia of Physics*, volume III/3. Springer-Verlag, Berlin, 1965.

[21] C.-C. Wang and C. Truesdell. *Introduction to Rational Elasticity*. Noordhoff, Leyden, 1973.

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